Factor Models 2: Factor Models as State Space Models and the Dynamic Factor Model

Advanced Time Series Econometrics

Scottish Graduate Programme in Economics

Readings

- Ghysels and Marcellino (G+M) textbook coverage of DFMs is scattered across more than one chapter
- G+M in chapter 11 (11.2.4 and 11.5.2): DFM as a state space modek
- G+M chapter 13.2 has DFM as a Big Data method
- Tsay textbook does not cover DFM
- Lecture slides also can count as "reading" a reading for these topics.
- Recent excellent (but more advanced) reading:
- Stock and Watson (2016) "Factor Models and Structural Vector Autoregressions in Macroeconomics"
- Chapter 8 in Handbook of Macroeconomics, available on Mark Watson's Princeton University website

Factor Models as State Space Models

- Last week factors were either known (market model) or replaced by Principal Components
- But they dan be treated as unobserved states
- Remember Normal Linear State Space model:

$$y_t = W_t \delta + Z_t \beta_t + \varepsilon_t$$

- State equation:

$$\beta_{t+1} = D_t \beta_t + u_t$$

- y_t contains dependent variable(s)
- If we set $y_t = r_t$, $D_t = 0$
- $W_t = 1$ and, thus, δ is intercept (α in our factor model notation)
- Z_t is the factor model's β (factor loadings)
- β_t are the factors (f_t)

Factor Models as State Space Models

- Static factor model is state space model
- Econometric theory of state space models (first lecture) holds here
- Kalman filtering and smoothing methods for estimation
- Information criteria for selecting models
- In **R** there is the `DFM` function from the `dfms` package, which uses state space methods
- To estimate a static factor model, we can use the `factanal()` command
- The `ICr()` function (also from the `dfms` package) provides additional information criteria for selecting the number of factors

Identification in Factor Models

- Remember Static Factor Model is

$$r_t = \alpha + \beta f_t + \varepsilon_t \qquad (*)$$

- When treating as state space model β,f_t and ε_t are not observed
- How can we estimate 3 separate unobserved things using one thing r_t?
- Identification is word we use for this
- Factor model is not identified without further restrictions
- Previously we have implicitly used identification assumptions
- E.g. in PCA said $w_i' w_i = 1$ and PCs uncorrelated with each other
- These were identification restrictions

Identification in Factor Models

- We previously made other assumptions which helped identification
- $cov(\varepsilon_t) = D$ where D is diagonal matrix
- Intuition: ε_t is idiosyncratic (ε_{it} is error specific to asset i, uncorrelated with other assets)
- But this is not enough
- Static factor model in (*) is equivalent to

$$
\begin{aligned}
r_t &= \alpha + \beta P^{-1} P f_t + \varepsilon_t \\
r_t &= \alpha + \beta^* f_t^* + \varepsilon_t
\end{aligned}
$$

  for any matrix P

- Equivalent model has new factors f_t^* and new factor loadings β^*

Example: Identification in Factor Models

- There are standard ways of identifying factor models
- E.g. assume $\Sigma_f = I$
- E.g. $\beta_1 = 1$
- But other restrictions often imposed on β to give economically-meaningful factors (as well as identification)
- E.g. r_t contains GDP growth for many countries around the world
- For illustration assume two regions: OECD countries (with $Noecd$ of them) and non-OECD countries ($N - Noecd$ of them)
- OECD countries ordered first

Example: Identification in Factor Models

- Consider β structured as:

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & . & . \\ . & \beta_{Noecd,2} & 0 \\ . & 0 & \beta_{Noecd+1,3} \\ . & 0 & . \\ . & 0 & . \\ \beta_{N1} & 0 & \beta_{N,3} \end{bmatrix}$$

- First column is unrestricted
- Second column is unrestricted for OECD countries, zeros otherwise
- Third column unrestricted for non-OECD countries, zeros otherwise

Example: Identification in Factor Models

- f_1 will load on all countries
- I.e. f_1 will be regressor in each of the N regressions
- But f_2 will only load on OECD countries
- f_3 will only load on non-OECD countries
- f_1 is world factor (e.g. world business cycle)
- f_2 is OECD factor
- f_3 is non-OECD factor

Identification in Factor Models

- Many research papers give factors an economically-meaningful interpretation in this way
- Factor loadings can be restricted so as to give:
- E.g. world, regional factors, etc.
- E.g. employment growth in different industries in Canadian provinces
- Canadian factor, provincial factor and industry-specific factor
- E.g. stock markets: financial services factor, computer sector factor, mining sector factor, etc.
- In Dynamic Factor Models similar strategies for identification can be used (see Ghysels and Marcellino 11.5.2)

Example: Static Factor Model

- Same returns data for IBM, HPQ, INTC, JPM and BAC as used with PCA
- To replicate some of the exercises, see the **R** code available on Tsay's website: faculty.chicagobooth.edu/ruey-s-tsay/research/multivariate-time-series-analysis-with-r-and-financial-applications
- Table on next slides contain output produced for maximum likelihood
- Automatically decides to retain 2 factors
- Automatically estimates factor loading matrix

Static Factor Results (Maximum Likelihood)		
Factor	Eigenvalue	Proportion
1	1.80	0.60
2	1.19	0.40

Factor Loadings		
Variable	f_1	f_2
IBM	0.33	0.53
HPQ	0.35	0.67
INTC	0.34	0.65
JPM	0.73	0.19
BAC	0.96	-0.11

Example: Discussion of Results

- General findings similar to those with PCA
- Decision to retain 2 factors same as scree plot PCA
- This implicitly set "proportions" for 3, 4 and 5 factors to zero
- Hence proportions for first two factors scaled up relative to PCA results

Example: Discussion of Results

- Remember with PCA Tsay argued results led to market and industrial component
- Market component: average of stock returns for all companies
- Industrial component: difference between computer stocks and bank stocks
- Here we cannot directly see if the factors are constructed in this way
- However, we can say the following:
- First factor loads more heavily on banks (JPM and BAC)
- Second factor loads more heavily on IBM, HPQ, INTC (with little weight to banking stocks)

Dynamic Factor Models (DFMs)

- In finance, static factor model often used
- In macroeconomics, DFMs more common
- Macroeconomic variables often persistent, static assumption that ε_t uncorrelated over time inappropriate
- Ghysels and Marcellino offers some coverage of DFMs
- Tsay's textbook does not cover
- Remember an advanced reading is: Stock and Watson (2016)
- This reading also links DFMs with structural VARs and Factor augmented VARs (FAVARs)

Dynamic Factor Models

- The `DFM()` function allows for the estimation of DFMs of the form:

$$
\begin{aligned}
y_t &= Pf_t + Qx_t + u_t \\
f_t &= Rw_t + A_1 f_{t-1} + .. + A_p f_{t-p} + v_t \\
u_t &= C_1 u_{t-1} + .. + C_q u_{t-q} + \varepsilon_t
\end{aligned}
$$

- ε_t is i.i.d. $N(0, \Sigma_\varepsilon)$
- v_t is i.i.d. $N(0, \Sigma_v)$
- y_t is $N \times 1$ vector of dependent variables
- f_t is $m \times 1$ vector of factors
- x_t and w_t are n_x exogenous variables
- $P, Q, R, A_1, .., A_p, C_1, .., C_q$ are all matrices of parameters to be estimated
- P are the factor loadings
- Note x_t and/or w_t could contain intercept

Dynamic Factor Models

- This is a very flexible specification (not identified) and can be hard to estimate (hard to achieve convergence)
- Usually you will work with restricted version
- For same reasons as in static factor model, usual to assume Σ_ε is diagonal
- Default settings typically assume Σ_ε is diagonal, $\Sigma_v = I$ and $A_1, .., A_p, C_1, .., C_q$ are diagonal matrices (see `help(DFM)` in **R**)
- Others possible (but gets harder to estimate, especially if N is large)
- See 11.5.2 of Ghysels and Marcellino for another example

Econometric Estimation of Dynamic Factor Models

- What about econometrics?
- This is a state space model and standard state space methods can be used
- Information criteria used to make specification choices (e.g. choose m, p, q)
- However computationally difficult when N is large
- Notice that, in computer tutorials and empirical examples, we never use large N, as it is computationally expensive
- Bayesian methods popular

Econometric Estimation of Dynamic Factor Models

- Ghysels and Marcellino (chapter 13.2) discuss two other estimators (computationally less demanding than state space methods)
- First uses PCA methods to estimate factors
- But PCA is static method which is drawback in DFM (but can show resulting estimates are consistent under some assumptions). More discussion of this below.
- Second is "three pass regression filter" which surmounts this problem
- I will not provide details (too difficult for MSc level course)

Properties of Dynamic Factor Models

- Similar to static factor model, but more flexible
- Idiosyncratic errors, u_t, have AR structure (useful for modelling persistence in macroeconomic variables)
- Factors, f_t, have VAR structure so can be persistent
- E.g. if f_{1t} captures "world business cycle" might expect it to evolve gradually over time
- Static factor model assumption that $cov\left(f_{1t}, f_{1t-1}\right) = 0$ may be bad
- But DFM allows for $cov\left(f_{1t}, f_{1t-1}\right) \neq 0$

Special Cases of Dynamic Factor Models

- Baseline: General case DFM with VAR errors
- Special cases are:
- DFM has $q = 0$
- Static factor model with VAR errors has $p = 0$
- Static factor model has $p = q = 0$
- VAR errors has $m = 0$ (no factors) but $q > 0$
- Seemingly unrelated regressions model has $m = q = p = 0$

Empirical Tips with DFMs

- Often including both $q > 0$ and $p > 0$ too flexible
- Having $p > 0$ enough to "clean up" any persistence in the data
- e.g. persistence in the data due to common factors
- Or, if N is small, let x_t included lagged dependent variables (i.e. a VAR)
- This enough to clean up persistence in data in many cases

Empirical Tips with DFMs

- PCA methods can also be used with DFMs (alternative to state space methods)
- E.g. assume $q = 0$ then DFM:

$$\begin{aligned} y_t &= Pf_t + Qx_t + u_t \\ f_t &= Rw_t + A_1 f_{t-1} + .. + A_p f_{t-p} + v_t \end{aligned}$$

- substitute second equation into first:

$$\begin{aligned} y_t &= P\left(Rw_t + A_1 f_{t-1} + .. + A_p f_{t-p} + v_t\right) f_t + Qx_t + u_t \\ &= R^* w_t + A_1^* f_{t-1} + .. + A_p^* f_{t-p} + Qx_t + u_t^* \end{aligned}$$

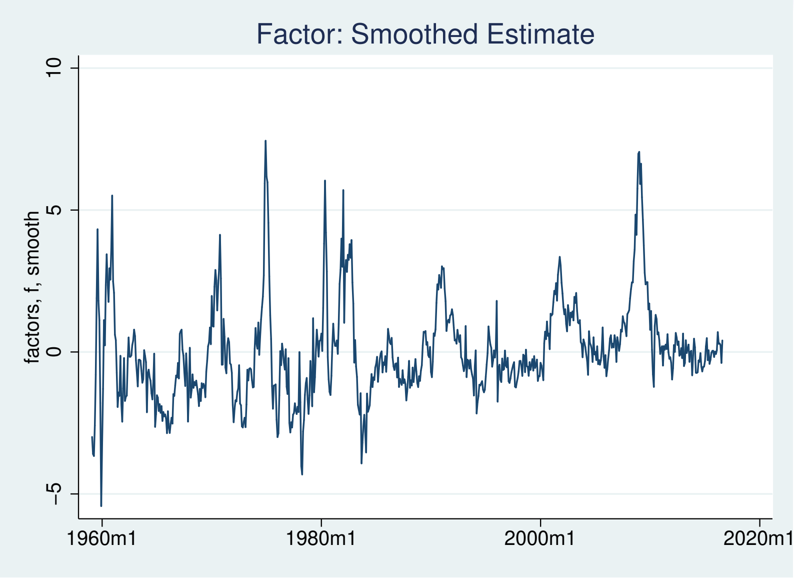
- $R^* = PR, A_1^* = PA_1$, etc.
- Replace $f_{t-1}, .., f_{t-p}$ by PC estimates
- Estimate a multivariate regression

Example: US Macroeconomic Data

- US monthly macroeconomic variables from January 1959 through August 2016
- FRED-MD data for 10 major macroeconomic aggregates reflecting range of concepts:
- Income, wages, labour market, housing starts, money, short and long term interest rates, etc.
- In table on next slide can see which variables I have chosen
- For precise definition of each variable see FRED-MD website
- Transformed to stationarity as recommended by FRED-MD
- One factor
- One lag in the errors ($q = 1$)
- One lag for the factors ($p = 1$)

Example: Coefficient Estimates in DFM

Variable	Intercept		Coeff. on Factor		Coeff. on Lag of Error or Factor	
	Estimate	P-val.	Estimate	P-val.	Estimate	P-val.
Factor	—	—	—	—	0.83	0.00
Income	0.003	0.00	−0.0001	0.00	−0.18	0.00
Ind. Prod.	0.002	0.00	−0.003	0.00	0.12	0.01
Unemp.	−0.003	0.84	0.058	0.00	−0.23	0.00
Employment	0.001	0.00	−0.001	0.00	−0.33	0.00
Wages	40.71	0.00	−0.13	0.00	0.95	0.00
House starts	7.22	0.00	−0.03	0.00	0.97	0.00
Money	0.002	0.00	0.000	0.55	0.61	0.00
Tbill_1yr	−0.003	0.91	−0.07	0.00	0.31	0.00
Tbill_10yr	0.005	0.00	−0.03	0.00	0.28	0.00
Stock mkt.	0.005	0.00	−0.001	0.44	0.24	0.00



Factor: Smoothed Estimate
factors, f, smooth
10
5
0
−5
1960m1
1980m1
2000m1
2020m1

Example: Interpretation of Results

- Most important results usually for factor loadings
- Coefficients on factors are significant for all variables except money and stock market variables
- Coefficient on factor is negative for each variable except for unemployment
- Factor = information common to all variables which may or may not have easy economic interpretation
- Perhaps = state of the economy
- Note: Factor gets very large (positive) when financial crisis hits
- Also just after OPEC oil shock, recession of early 1980s, dotcom bubble, etc.
- Negative coefficients mean all variables (except unemployment) go down in these times
- But unemployment goes up in bad times
- Our factor is measuring this

Example: Interpretation of Results

- Remember identification (can multiply factor and factor loads both by minus one and get same model)
- So, multiplying factor by -1, could get new factor which is "good times" factor
- Preceding discussion about coefficient on factor in equation for each variable
- For other parameters:
- An intercept is usually significant
- Factor equation indicates importance of DFM (as opposed to static factor model):

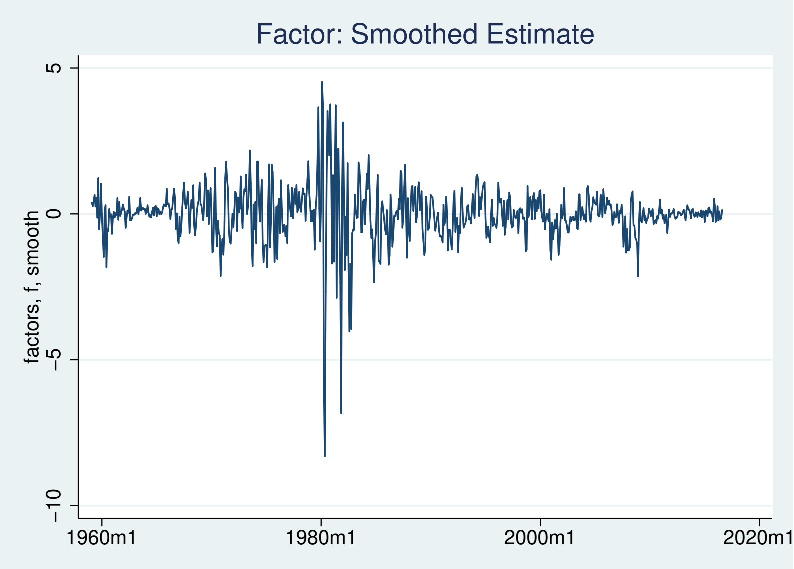
$$f_t = 0.83 f_{t-1} + v_t$$

- Evidence for AR(1) errors in other equations

Example: Interpretation of Results (Static Factor Model)

- I repeat the previous analysis, except using the Static Factor model
- I will not discuss all the parameter estimates
- Next slide presents factor estimate
- In DFM factor was AR(1) with coeff. $= 0.83$ (quite persistent)
- Static factor model imposes AR coeff $= 0$
- Factor uncorrelated over time
- Estimate factor very erratic, hard to interpret
- For DFM: AIC and BIC are: -27029.04 and -26842.98
- For SFM: -22762.94 and -22626.8
- Choose DFM over SFM

Example: Smoothed Estimate of the Factor using Static Factor Model



Factor Augmented VARs

- Hot topic
- Combining Factor methods with VARs in Big Data applications
- y_t is $N \times 1$ vector of observed variables
- Want to build a VAR (e.g. for impulse response analysis, etc.), but N is large
- Bayesian methods for large VARs exist, but what if you are not Bayesian?
- Isolate a few variables of interest (e.g. interest rate, unemployment rate and inflation)
- E.g. impulse response of monetary shock relates to interest rate and this is your main focus
- y^* are these core variables of interest
- y^o are the other variables

Factor Augmented VARs

- Build factor model for other variables which includes core variables on right hand side:

$$y_{it}^{o} = P_i f_t + \gamma_i y_t^{*} + \varepsilon_{it} \qquad (1)$$

- where i indicates individual variables (and P_i factor loadings for equation i)
- Then VAR for core variables and the factors:

$$\begin{pmatrix} f_t \\ y_t^{*} \end{pmatrix} = \Phi_1 \begin{pmatrix} f_{t-1} \\ y_{t-1}^{*} \end{pmatrix} + .. + \Phi_p \begin{pmatrix} f_{t-p} \\ f_{t-p} \end{pmatrix} + \varepsilon_t^{f}$$

- Equation (1) distills all the information in y^o into a few factors
- Equation (2) is a small VAR with only core variables and these few factors
- Pioneering paper was: Bernanke, Boivin and Eliasz (2005). "Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR) Approach," Quarterly Journal of Economics.

FAVARs

- FAVARs are state space models
- Econometric estimation using state space methods (Kalman filter, etc.)
- Or can use two step method
- Step 1: Use PCA to get $\widehat{f}_t$ (estimate of f_t)
- Step 2: Build a VAR for y_t^* and $\widehat{f}_t$ using methods taught in Econometrics 2

Summary

- Factor models are state space models
- Econometric estimation and specification issues same as for state space models
- Dynamic Factor model is popular in macroeconomics
- Extensions such as Factor-augmented VAR popular