Nonlinear Time Series Models: Structural Breaks and Threshold Autoregression

Advanced Time Series Econometrics

Scottish Graduate Programme in Economics

Introduction

- Regressions, ARMA models and VARs are all linear
- Illustrate with a regression:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

- Straight line relationship between y and x
- β is marginal effect of x on y
- Marginal effect is same regardless of value of x
- Nonlinear models = anything that is not linear
- Marginal effect may be different depending on what value of x is
- E.g. quadratric relationship

$$y_t = \alpha + \beta_1 x_t + \beta_2 x_t^2 + \varepsilon_t$$

- Marginal effect is $\beta_1 + 2\beta_2 x_t$

Types of Nonlinearity

- In macroeconomics and finance certain types of nonlinearities of particular interest
- Abrupt change: at a point in time β abruptly changes
- Structural break models
- E.g. financial crisis fundamentally altered the financial markets and relationships between variables changed
- Gradual change: β is gradually changing over time
- Time varying parameter (TVP) models
- E.g. financial liberalizations throughout the 1980s and 1990s gradually changed the relationships between financial variables

Types of Nonlinearity

- Regime switching models: β is different in different regimes
- E.g. Regime = state of the business cycle
- Expansion and Recession might be two regimes
- E.g. β measures the effectiveness of monetary policy
- Measures how much inflation increases when money supply increased
- Printing money in recessionary times might not cause inflation ($\beta = 0$)
- Printing money when economy is booming will (β large)

Types of Nonlinearity

- So far talked about nonlinearities in β
- Nonlinearities in error variance (σ^2)
- E.g. high volatility regime versus low volatility regime in stock markets
- E.g. Risk on/Risk off behaviour of financial market participants
- E.g. Great Moderation of Business Cycle
- In early 1980s volatility of many macro variables dropped

Why Worry About Nonlinearities?

- Strong empirical evidence it exists in many (most?) macroeconomic and financial time series
- Macro variables: Stock and Watson (1996) "Evidence on Structural Instability in Macroeconomic Time Series Relations" *Journal of Business and Economic Statistics*
- Financial Variables: Ang and Bekaert (2002) "Regime Switches in Interest Rates" *Journal of Business and Economic Statistics*
- Many other papers present similar findings
- If nonlinearities exist linear models are mis-specified
- Linear researcher gives wrong policy advice/bad forecasts, etc.
- E.g. How much does inflation increase when money supply increase?
- Linear methods give average estimate over expansions and recessions
- Average over time where $\beta = 0$ and β large might give β fairly large

The Econometrics of Nonlinear Time Series Models

- This lecture goes through various models of breaks/regime switching/TVP
- Focus is on the models, their properties and how to use them in practice
- Only a little about theory of estimation and hypothesis testing
- Maximum likelihood or Bayesian methods or least squares typically done

The Econometrics of Nonlinear Time Series Models

- Stata will produce usual likelihood based model choice methods including:
- Information criteria, tests of significance of individual parameters and likelihood ratio tests
- Reading 1: Chapters 9 and 10 of Ghysels and Marcellino
- Reading 2: Chapter 4 of Tsay

Models with Structural Breaks

- Use AR(1) model to illustrate ideas
- Extensions to AR(p) easy (just add more lags)
- And usually want to include an intercept (results below do)
- Case of structural breaks in regression coefficients is easy (just replace y_{t-1} by x_t)
- A simple structural break model:

$$y_t = \begin{cases} \rho_1 y_{t-1} + \varepsilon_t \text{ if } t \leq \tau \\ \rho_2 y_{t-1} + \varepsilon_t \text{ if } t > \tau \end{cases}$$

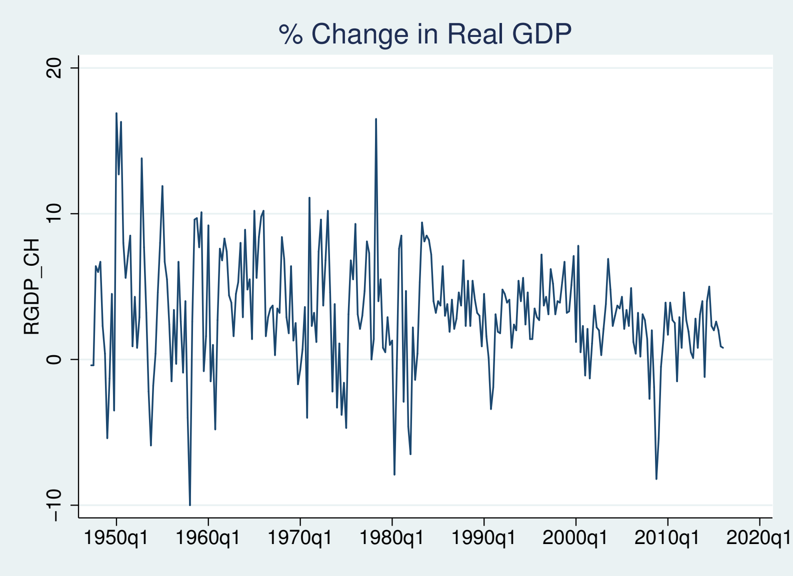
- Different AR(1) models before or after *break date* τ
- For now assume break date known
- Note: extension to multiple breaks is straightforward

Example: Breaks in US GDP growth

- Data: Real Gross Domestic Product, Percent Change from Preceding Period, Quarterly, Seasonally Adjusted Annual Rate
- Plotted on next graph
- Can write structural break model as

$$y_t = \rho_1 y_{t-1} + \gamma x_{t-1} + \varepsilon_t$$

  where $x_t = D_t y_t$
- Dummy variable: $D_t = 1$ if $t > \tau$, else $D_t = 0$
- If $t \leq \tau$ then coefficient on y_{t-1} is ρ_1
- If $t > \tau$ coefficient on y_{t-1} is $\rho_1 + \gamma$ (same as ρ_2 on previous slide)


% Change in Real GDP
RGDP_CH
20
10
0
−10
1950q1
1960q1
1970q1
1980q1
1990q1
2000q1
2010q1
2020q1

Example: Breaks in US GDP growth

- OLS estimates of the AR(1) model give fitted regression line

$$y_t = 2.04 + 0.37 y_{t-1}$$

- Adding x_{t-1} using a break date $\tau = 1983Q1$ (beginning of Great Moderation of the Business Cycle?) gives

$$y_t = 2.06 + 0.38 y_{t-1} - .02 x_{t-1}$$

- Two fitted regressions look pretty similar, so maybe no break in $1983Q1$?
- Can test this: if $\gamma = 0$, same AR model before/after break
- T-stat for $H_0 : \gamma = 0$ is -0.20 with p-value 0.84
- Accept H_0 there is no break in $1983Q1$
- Another way to check is information criteria
- AR(1) gives AIC and BIC of 1495 and 1502
- AR(1) with break gives AIC and BIC of 1496 and 1507
- Choose AR(1) over model with break

Example: Breaks in US GDP growth

- Preceding assumed same error variance before and after break
- What if model is:

$$y_t = \begin{cases} \rho_1 y_{t-1} + \varepsilon_{1t} \text{ if } t \leq \tau \\ \rho_2 y_{t-1} + \varepsilon_{2t} \text{ if } t > \tau \end{cases}$$

- $var\left(\varepsilon_{1t}\right) = \sigma_1^2$ and $var\left(\varepsilon_{2t}\right) = \sigma_2^2$
- Can estimate this model by simply dividing the data into two parts
- Part 1: all observations with $t \leq \tau$
- Part 2: all observations with $t > \tau$

Example: Breaks in US GDP growth

- Divide the data into pre- and post-1983 samples
- Two OLS regressions yield:
- Fitted regression line with pre-1983 sample:

$$y_t = 2.41 + 0.34 y_{t-1}$$

- $s^2 = 20.85$
- Post-1983:

$$y_t = 1.46 + 0.49 y_{t-1}$$

- $s^2 = 5.14$
- Looks like break in error variance:
- Great Moderation of the Business Cycle: s^2 much lower after 1983
- But how confident statistically?

Example: Breaks in US GDP growth

- Three models: AR(1), AR(1) with break in mean (i.e. ρ changes) and AR(1) with break in both mean and error variance
- Use information criteria (IC) to choose between them
- Note: IC for model with break in mean and variance just add together ICs from pre- and post-break regressions
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	AIC	BIC
AR(1)	1494.74	1501.98
AR(1) with break in mean	1496.70	1507.55
AR(1) using pre-break data	842.18	848.10
AR(1) using post-break data	592.56	598.32
AR(1) with break in mean and variance	1434.74	1446.43

- AR(1) with breaks in mean and variance has lowest IC's – it is best model

Chow Test

- General test used with regression or AR (here use for break)
- Do two parts of your sample have same regression line?
- $H_0 : \rho_1 = \rho_2, \sigma_1^2 = \sigma_2^2$
- Steps in Chow Test using break date τ
- Estimate AR(1) model using entire sample and get Sum of Squared Residuals (SSR_0)
- Estimate AR(1) model using $t \leq \tau$ sample and get SSR_1
- Estimate AR(1) model using $t > \tau$ sample and get SSR_2

Chow Test

- Chow test statistic is:

$$Chow = \frac{\frac{SSR_0 - (SSR_1 + SSR_2)}{k}}{\frac{SSR_1 + SSR_2}{T_1 + T_2 - 2k}}$$

- $T_1 =$ number of observations in $t \leq \tau$ sample
- $T_2 =$ number of observations in $t > \tau$ sample
- $k =$ number of explanatory variables (plus intercept) in model
- In AR(1) $k = 2$
- If H_0 is true $Chow \sim F\left(k, T_1 + T_2 - 2k\right)$
- Get critical value from F statistical tables

Example: Breaks in US GDP growth

- In our example $k = 2,\ T_1 = 132,\ T_2 = 142$
- $SSR_0 = 3640.1, SSR_1 = 2940.6, SSR_2 = 667.6$
- Plugging these in we get $Chow = 1.19$
- 5% critical value from $F\left(2, 270\right)$ is 3.09
- Test statistic less than critical value so accept H_0
- In contrast to IC results, this indicates no break
- Classical hypothesis testing: only reject H_0 if overwhelming evidence against it
- BIC is (approximately) proportional to log of probability model generated data
- ratio of exp of BICs for two models = relatively probabilities of each model
- Hypothesis tests/ICs have different interpretations so sometimes they give different results

The Threshold Autoregressive (TAR) Model

- TAR models have same form as structural break models
- But different variable triggers regime change

$$y_t = \begin{cases} \rho_1 y_{t-1} + \varepsilon_{1t} \text{ if } z_t \leq \tau \\ \rho_2 y_{t-1} + \varepsilon_{2t} \text{ if } z_t > \tau \end{cases}$$

- $var\left(\varepsilon_{1t}\right) = \sigma_1^2$ and $var\left(\varepsilon_{2t}\right) = \sigma_2^2$
- τ is called the threshold

The TAR Model

- z_t (the trigger) can be any exogenous or lagged dependent variable
- Common choice: $z_t = y_{t-d}$
- $d =$delay parameter
- Must have $d > 0$
- Having y_t being dependent variable and also trigger runs into endogeneity problems
- z_t could be some other variable (e.g. change in oil price)
- Structural break model is special case of TAR with $z_t = t$

The TAR Model

- Assume τ (and d) known (talk about estimating them later)
- Econometrics exactly the same as structural break model (with known break data) covered previously
- E.g. GDP growth and $z_t = y_{t-1}$ and $\tau = 0$
- Two regimes: recession ($y_{t-1} \leq 0$) and expansion ($y_{t-1} > 0$)
- Approach 1: TAR with change in mean
- Create $D_t =$ a dummy variable $= 1$ for recessions, 0 for expansions
- Create $x_t = D_t y_t$
- Add x_{t-1} as explanatory variable to AR(1) model
- Approach 2: TAR with change in mean and variance
- Run separate AR(1) models for recession and expansion

Example: Regime Switches in US GDP growth

- GDP growth data
- $z_t = y_{t-1}$ and $\tau = 0$
- Table of information criteria in same format as for structural breaks
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	AIC	BIC
AR(1)	1494.74	1501.98
TAR(1) with change in mean	1496.54	1507.39
AR(1) using $y_{t-1} \geq 0$	1241.28	1248.19
AR(1) using $y_{t-1} < 0$	247.96	251.38
TAR(1) with change in mean and variance	1489.24	1499.57

- Similar to structural break results

Example: Regime Switches in US GDP growth

- Little evidence that recessionary and expansionary regimes have different AR coefficients
- But there is evidence that error variances differ between recessions and expansions
- Larger error variances in recessions
- In substantive application, would consider longer lag length, different thresholds, triggers and delays
- Or even multiple regimes
- E.g. 4 regimes: 1) recessions, 2) recovery from recessions 3) normal times and 4) over-heating
- For financial applications, might want structural break or regime switching in volatility
- E.g. Tsay, page 182-183 has threshold GARCH model
- Asymmetric responses to positive and negative return to an asset.

What if Break Dates or Thresholds Unknown?

- Have now gone through basic ideas of structural break and TAR class of regime switching models
- But always assumed τ (and z_t and d) known
- The idea underlying estimation is simple:
- Estimate model for every possible choice for τ (and z_t and d if relevant)
- Choose value of τ with highest value for likelihood (MLE)
- Choose value for τ with lowest sum of squared residuals (least squares estimator)
- May want to restrict possible values for τ to make sure regimes contain a minimum number of observations
- E.g. break does not occur in first or last 5% of the observations

What if Break Dates or Thresholds Unknown?

- You have to program in Stata to do estimation in this way using loops
- Loops are commands of the form:
- "for each [value of τ] in [a grid of possible values] [estimate the TAR]"
- And you have to specify things in [...]
- I won't ask you to estimate τ in computer labs

What if Break Dates or Thresholds Unknown?

- With multiple breaks and other parameters can become computationally demanding
- E.g. Monthly data for 40 years ($T = 480$)
- 3 regimes (2 thresholds) TAR model (τ_1, τ_2) with 12 values for delay ($d = 1, .., 12$)
- On the order of $12 \times 480^2 = 2764800$ TARs to estimate
- Can reduce this somewhat by imposing restrictions (e.g. $\tau_2 > \tau_1$ and each regime must contain at least 40 observations)
- But still a lot of TARs to estimate

Testing for Nonlinearity

- Can always use information criteria to choose between different nonlinear time series models
- Or choose between nonlinear and linear
- What about formal hypothesis testing methods?
- If τ is known can use Chow test
- But what is τ is unknown?
- Tests break into two groups:
- 1. Test in a particular model (e.g. testing TAR/structural break versus AR)
- 2. general test for departures from linearity

Testing for Thresholds/Structural Breaks

- Same issues hold for TAR and structural break model, illustrate with former
- Bottom line: econometric theory hard and hard to do these tests in Stata (need to program)
- Consider hypothesis test for whether TAR is preferred to AR
- $$y_t = \begin{cases} \rho_1 y_{t-1} + \varepsilon_{1t} \text{ if } z_t \leq \tau \\ \rho_2 y_{t-1} + \varepsilon_{2t} \text{ if } z_t > \tau \end{cases}$$
- $H_0 : \rho_1 = \rho_2, \sigma_1^2 = \sigma_2^2$
- $H_1 : \rho_1 \neq \rho_2, \sigma_1^2 \neq \sigma_2^2$

Testing for Thresholds/Structural Breaks

- But what about τ? H_0 says nothing about it as it does not appear in AR model
- Jargon: Nuisance parameter unidentified under the null or Davies' Problem
- Complicates the econometric theory of deriving asymptotic distribution of test statistic
- Also have to program in Stata
- Beyond scope of this course, but see page 211 of Tsay textbook if interested
- Davies' problem make several nonlinear tests complicated
- Tsay, page 212 describes another TAR test which does not run into Davies' problem (but complicated in other ways)

Testing for Thresholds/Structural Breaks

- A simple test statistic for TAR can be obtained using the sums of squared residuals for the linear (SSR_0) and TAR (SSR_1) relative to estimated error variance from TAR (s^2)
- But SSR_1 and s^2 depend on choice for threshold so write them as $SSR_1(\tau)$ and $s^2(\tau)$
- Hansen test statistic is based on

$$H(\tau) = \frac{SSR_0 - SSR_1(\tau)}{s^2(\tau)}$$

- Test statistic is $H(\tau^*)$ where τ^* is chosen to make the test statistic as large as possible

Testing for Thresholds/Structural Breaks

- To do hypothesis testing you need test statistic and critical value
- We have a test statistic $H(\tau^*)$, what is critical value?
- To get critical value you need to know distribution of test statistic
- Distribution of $H(\tau^*)$ is not know analytically
- Need to use numerical methods
- This is not done in standard software like Stata

Tests for Departures from Linearity

- Several tests exist to check whether linear model is adequate without specifying alternative model
- I will discuss one of these which can be easily done in Stata
- Intuition: if linear model is okay, then its errors should be white noise
- Estimate linear model, calculate residuals and test if they are white noise
- If not, then better consider a nonlinear model such as structural break or TAR
- Note: sometimes looking at patterns in residuals can suggest what form of nonlinearity.
- E.g. if large residuals bunch at one point in time might indicate break

RESET Test

- Can be used with regression or AR(p) models, illustrate with AR(1):

$$y_t = \rho y_{t-1} + \varepsilon_t$$

- Obtain OLS residuals $\widehat{\varepsilon}_t$ and their sum of squares, SSR_0
- If linear model is correct, these residuals should be white noise (unpredictable)
- Run a second OLS regression of:

$$\widehat{\varepsilon}_t = \phi_1 \widehat{\varepsilon}_{t-1} + \phi_2 \widehat{\varepsilon}^2_{t-1} + v_t$$

  and get sum of squared residuals, SSR_1
- Note: can have higher powers (e.g. $\widehat{\varepsilon}^3_{t-1}$) on right hand side of second regression

RESET Test

- $H_0 : \phi_1 = \phi_2 = 0$
- If H_0 true then $SSR_1 = SSR_0$
- If H_1 true then $SSR_1 < SSR_0$
- This intuition forms basis of RESET test
- Test statistic

$$RS = \frac{\left(SSR_0 - SSR_1\right)/g}{SSR_1/\left(T - p - g\right)}$$

- Note: p is AR lag length and and g is highest power on $\widehat{\varepsilon}_{t-1}$ in second regression minus one
- If H_0 is true, $RS \sim F\left(g, t - p - g\right)$
- Get critical values from F-statistical tables

Practical Recommendations for Model Choice with Nonlinear Time Series Models

- Often researcher faces questions
- Is a time series model linear or nonlinear?
- If nonlinear what type of nonlinearity?
- Sometimes economic theory and knowledge about the economy can help answer
- If not, then experiment:
- Try some generic test (like RESET or others in textbook)
- If any indication of nonlinearity, look at IC's for a few different nonlinear models
- E.g. TAR (different choices for z_t, d and τ), structural break models (different choices for τ)
- Choose specification with best IC

Other Threshold Autoregressive Models

- Several variants on the TAR model I will not go through
- STAR = smooth transition autoregressive model (see chapter 9 of textbook)
- e.g. TAR switches abruptly from one regime to another, STAR does this more gradually
- My models and examples involve switching between two regression models
- Can do very similar things with error variances
- E.g. Tsay (Chapter 3) has threshold GARCH model (TGARCH)
- GARCH involves equation for volatilities of asset returns, r_t

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \alpha_2 r_{t-1}^2$$

- TGARCH allows this to differ across regimes
- e.g. one GARCH equation if $r_{t-1} < \tau$, another if $r_{t-1} \geq \tau$

Summary

- In many (most?) macro and finance time series evidence of parameter change
- Often need nonlinear time series model to avoid mis-specification
- But what kind of nonlinearity?
- We have gone through variety of models in this lecture
- These feature abrupt break at specific time: Structural break
- Or abrupt change depending on what value of lagged dependent (or exogenous) variable is: TAR
- Lecture focusses on AR(1) but these models also work with AR(p), regressions, VARs or even volatilities
- But there are other models with other types of parameter change and some we will cover in next week's lecture