SCOTTISH GRADUATE PROGRAMME IN ECONOMICS

ECNM11060

Bayesian Econometrics

Popular Bayesian state space models in macroeconometrics

Introduction

- $\Rightarrow$ Macro researchers usually have dozens or hundreds of time series variables to work with
- $\Rightarrow$ This has led to large VARs
- $\Rightarrow$ Bayesian (hierarchical) priors used to surmount challenge of overfitting
- $\Rightarrow$ Instead of using prior shrinkage on parameters, why not compress/shrink the data itself
- $\Rightarrow$ Work with smaller, more parsimonious model, with compressed data
- $\Rightarrow$ This is the idea motivating factor models
- $\Rightarrow$ These are state space models so Bayesian inference straightforward

The static factor model

- ⇒ $\boldsymbol{y}_t$ is $M \times 1$ vector of time series variables
- ⇒ M is very large
- ⇒ Simplest static factor model reads as:

$$\boldsymbol{y}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda} \boldsymbol{f}_t + \varepsilon_t$$

- ⇒ $\boldsymbol{f}_t$ is $q \times 1$ vector of unobserved latent factors (where $q \ll M$)
- ⇒ Factors contain (common) information extracted from all the M variables
- ⇒ Same $\boldsymbol{f}_t$ occurs in every equation for y_{it} for $i = 1, ..., M$
- ⇒ But different coefficients ($\boldsymbol{\Lambda}$ is an $M \times q$ matrix of so-called factor loadings)

Identification issues in factor model

- ⇒ Note that restrictions are necessary to identify the model
- ⇒ Common to say ε_t is $\mathcal{N}(\mathbf{0}, \boldsymbol{D})$ where $\boldsymbol{D}$ is diagonal matrix
- ⇒ Implication: ε_{it} is pure random shock specific to variable i, co-movements in the different variables in $\boldsymbol{y}_t$ arise only from the factors
- ⇒ Note also that $\boldsymbol{\Lambda f}_t = \boldsymbol{\Lambda CC}^{-1}\boldsymbol{f}_t$ which shows we need identification restriction for factors too
- ⇒ Different models arise from different treatment of factors
- ⇒ Simplest is: $\boldsymbol{f}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_q)$
- ⇒ This can be interpreted as a state equation for “states” $\boldsymbol{f}_t$
- ⇒ Factor models are state space models so our MCMC tools for state space methods can be used

Dynamic factor models (DFMs) and factor augmented VARs (FAVARs)

The dynamic factor model (DFM)

- ⇒ In macroeconomics, usually need to extend static factor model to allow for the dynamic properties which characterize macroeconomic variables
- ⇒ A typical DFM:

$$\begin{aligned} y_{it} &= \lambda_{0i} + \boldsymbol{\lambda}_i' \boldsymbol{f}_t + \varepsilon_{it} \\ \boldsymbol{f}_t &= \boldsymbol{\Phi}_1 \boldsymbol{f}_{t-1} + \cdots + \boldsymbol{\Phi}_p \boldsymbol{f}_{t-p} + \boldsymbol{\eta}_t \end{aligned}$$

- ⇒ $\boldsymbol{f}_t$ is as for static model
- ⇒ $\boldsymbol{\lambda}_i$ is $q \times 1$ vector of factor loadings
- ⇒ Each equation has its own intercept, λ_{0i}
- ⇒ $\varepsilon_{it} \sim \mathcal{N}\left(0, r_i^2\right)$
- ⇒ $\boldsymbol{f}_t$ is VAR with $\boldsymbol{\eta}_t \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$

Replacing factors by estimates: Principal components

- ⇒ Proper Bayesian analysis of the DFM treats $\boldsymbol{f}_t$ as vector of unobserved latent variables
- ⇒ Before doing this, we note a simple approximation
- ⇒ The DFM has similar structure to regression model:

$$y_{it} = \lambda_{0i} + \widetilde{\boldsymbol{\lambda}}'_{0i}\boldsymbol{f}_t + \cdots + \widetilde{\boldsymbol{\lambda}}'_{pi}\boldsymbol{f}_{t-p} + \widetilde{\varepsilon}_{it}$$

- ⇒ If $\boldsymbol{f}_t$ were known we could use Bayesian methods for the multivariate Normal regression model to estimate or forecast with the DFM
- ⇒ Principal components methods can be used to approximate $\boldsymbol{f}_t$
- ⇒ Precise details of how principal components is done provided in many places

Treating factors as unobserved latent variables

- $\Rightarrow$ DFM is a Normal linear state space model so use Bayesian MCMC methods for state space models
- $\Rightarrow$ Conditional on the model's parameters, $\boldsymbol{\Sigma}, \boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p, \lambda_{0i}, \boldsymbol{\lambda}_i, r_i^2$ for $i = 1, .., M$, use e.g. Carter and Kohn algorithm to draw $\boldsymbol{f}_t$
- $\Rightarrow$ Conditional on the factors, measurement equations are just M Normal linear regression models
- $\Rightarrow$ Since ε_{it} is independent of ε_{jt} for $i \neq j$, posteriors for $\lambda_{0i}, \boldsymbol{\lambda}_i, r_i^2$ in the M equations are independent over i
- $\Rightarrow$ Hence, the parameters for each equation can be drawn one at a time (conditional on factors)
- $\Rightarrow$ Finally, conditional on the factors, the state equation is a VAR(p)
- $\Rightarrow$ Any of the priors for Bayesian VARs discussed before can be used

The factor augmented VAR (FAVAR)

$\Rightarrow$ DFMs are good for forecasting (extract all information in huge number of variables)

$\Rightarrow$ VARs are good for macroeconomic policy (e.g., impulse responses)

$\Rightarrow$ Why not combine DFMs and VARs together to get model which can do both?

$\Rightarrow$ FAVAR results

$\Rightarrow$ Bernanke, Boivin and Eliasz (2005, *QJE*) is pioneering paper

The FAVAR (cont.)

⇒ FAVAR modifies DFM by adding other explanatory variables:

$$y_{it} = \lambda_{0i} + \boldsymbol{\lambda}_i' \boldsymbol{f}_t + \gamma_i' \boldsymbol{r}_t + \varepsilon_{it}$$

⇒ $\boldsymbol{r}_t$ is $k_r \times 1$ vector of observed variables of key interest

⇒ For example, Bernanke, Boivin and Eliasz (2005) set $\boldsymbol{r}_t$ to be the Fed Funds rate (monetary policy instrument)

⇒ All other assumptions are same as for the DFM

⇒ Note: By treating $\boldsymbol{r}_t$ in this way, we can isolate a “monetary policy shock” and calculate impulse responses

The FAVAR (cont.)

⇒ FAVAR state equation extends DFM state equation to include $\boldsymbol{r}_t$:

$$\begin{pmatrix} \boldsymbol{f}_t \\ \boldsymbol{r}_t \end{pmatrix} = \widetilde{\boldsymbol{\Phi}}_1 \begin{pmatrix} \boldsymbol{f}_{t-1} \\ \boldsymbol{r}_{t-1} \end{pmatrix} + \cdots + \widetilde{\boldsymbol{\Phi}}_p \begin{pmatrix} \boldsymbol{f}_{t-p} \\ \boldsymbol{r}_{t-p} \end{pmatrix} + \widetilde{\boldsymbol{\eta}}_t$$

⇒ All assumptions are same as DFM with extension that $\widetilde{\boldsymbol{\eta}}_t \sim \mathcal{N}\left(\mathbf{0}, \widetilde{\boldsymbol{\Sigma}}\right)$

⇒ MCMC is very similar to that for the DFM and will not be described here

⇒ Similar ideas:

- → Normal linear state space algorithms can draw $\boldsymbol{f}_t$
- → Measurement equation is series of regressions (conditional on factors)
- → The state equation is a VAR (conditional on factors)

The TVP-FAVAR

⇒ With VARs: Began with constant parameter model

⇒ Then we said it is good to allow the VAR coefficients to vary over time: Homoskedastic TVP-VAR

⇒ Then we said good to allow for multivariate stochastic volatility: Heteroskedastic TVP-VAR

⇒ Can do the same with FAVARs

⇒ Note: Just as with TVP-VARs, TVP-FAVARs can be overparameterized and careful incorporation of prior information or the imposing of restrictions (e.g., only allowing some parameters to vary over time) can be important in obtaining sensible results

The TVP-FAVAR (cont.)

- ⇒ A TVP-FAVAR is just like a FAVAR but with t subscripts on parameters:

$$y_{it} = \lambda_{0it} + \boldsymbol{\lambda}_{it}' \boldsymbol{f}_t + \gamma_{it}' \boldsymbol{r}_t + \varepsilon_{it},$$

- ⇒

$$\begin{pmatrix} \boldsymbol{f}_t \\ \boldsymbol{r}_t \end{pmatrix} = \widetilde{\boldsymbol{\Phi}}_{1t} \begin{pmatrix} \boldsymbol{f}_{t-1} \\ \boldsymbol{r}_{t-1} \end{pmatrix} + \cdots + \widetilde{\boldsymbol{\Phi}}_{pt} \begin{pmatrix} \boldsymbol{f}_{t-p} \\ \boldsymbol{r}_{t-p} \end{pmatrix} + \widetilde{\boldsymbol{\eta}}_t$$

- ⇒ Allow each ε_{it} to follow univariate stochastic volatility (SV) process
- ⇒ $\mathbb{V}\left(\widetilde{\boldsymbol{\eta}}_t\right) = \widetilde{\boldsymbol{\Sigma}}_t$ has multivariate SV process of the form used in Primiceri (2005, *ReStud*)
- ⇒ Finally, for $i = 1, \ldots, M$, the coefficients $\lambda_{0it}, \boldsymbol{\lambda}_{it}, \gamma_{it}, \widetilde{\boldsymbol{\Phi}}_{1t}, \ldots, \widetilde{\boldsymbol{\Phi}}_{pt}$ are allowed to evolve according to random walks (i.e., state equations of the same form as in the TVP-VAR complete the model)
- ⇒ All other assumptions are the same as for the FAVAR

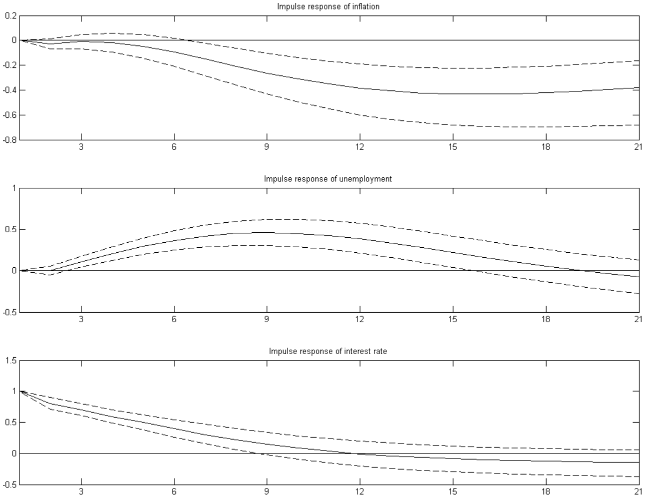
Bayesian inference in the TVP-FAVAR

- ⇒ I will not provide details of MCMC algorithm
- ⇒ Note only it adds more blocks to the MCMC algorithm for the FAVAR
- ⇒ These blocks are all of forms discussed in previous lectures
- ⇒ For example, error variances in measurement equations drawn using the univariate stochastic volatility algorithm of Kim, Shephard and Chib (1998, ReStud)
- ⇒ Multivariate stochastic volatility algorithm of Primiceri (2005, *ReStud*) can be used to draw $\widetilde{\boldsymbol{\Sigma}}_t$
- ⇒ The coefficients $\lambda_{0it}, \boldsymbol{\lambda}_{it}, \gamma_{it}, \widetilde{\boldsymbol{\Phi}}_{1t}, \ldots, \widetilde{\boldsymbol{\Phi}}_{pt}$ are all drawn using algorithm for Normal linear state space model

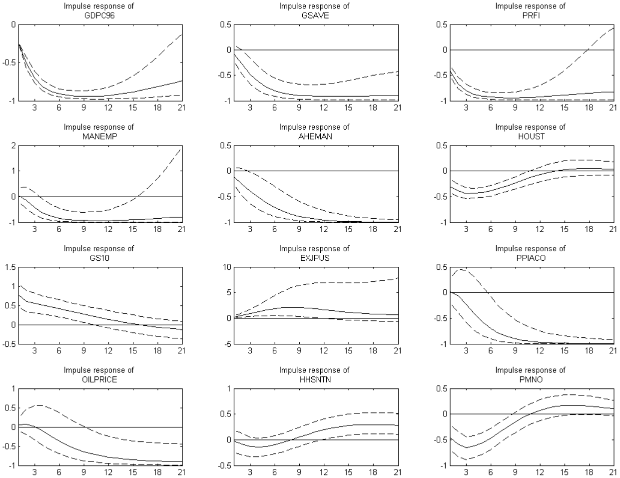
Empirical illustration of the FAVAR and TVP-FAVAR

- ⇒ 115 quarterly US macroeconomic variables spanning 1959Q1 through 2006Q3
- ⇒ Transform all variables to be stationary
- ⇒ What variables to put in $\boldsymbol{r}_t$?
- ⇒ Inflation, unemployment and the interest rate
- ⇒ FAVAR is same as VAR from previous lectures, but augmented with factors $\boldsymbol{f}_t$
- ⇒ We use 2 factors and 2 lags in state equation
- ⇒ Identify impulse responses (IRFs) to a monetary policy (MP) shock
- ⇒ Solid lines are posterior medians; dashed lines in some of following figures denote credible intervals

FAVAR: IRFs of main variables



FAVAR: IRFs of selected other variables



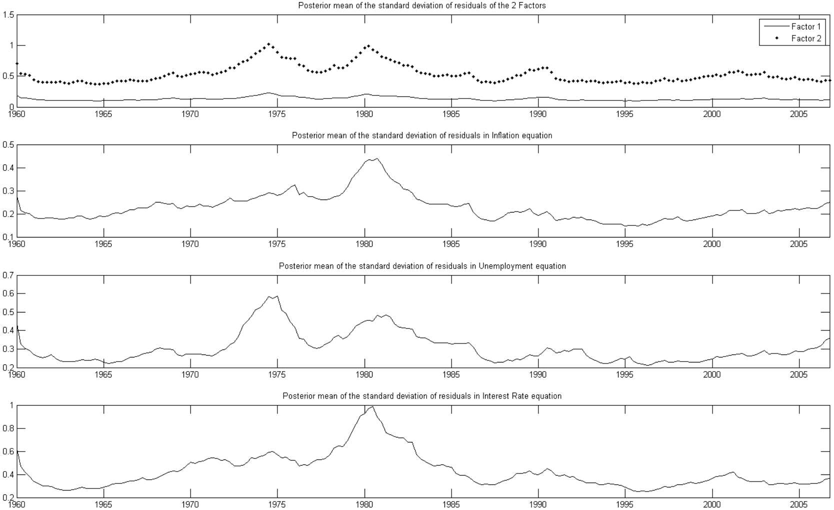
TVP-FAVAR: IRFs

⇒ Now TVP-FAVAR

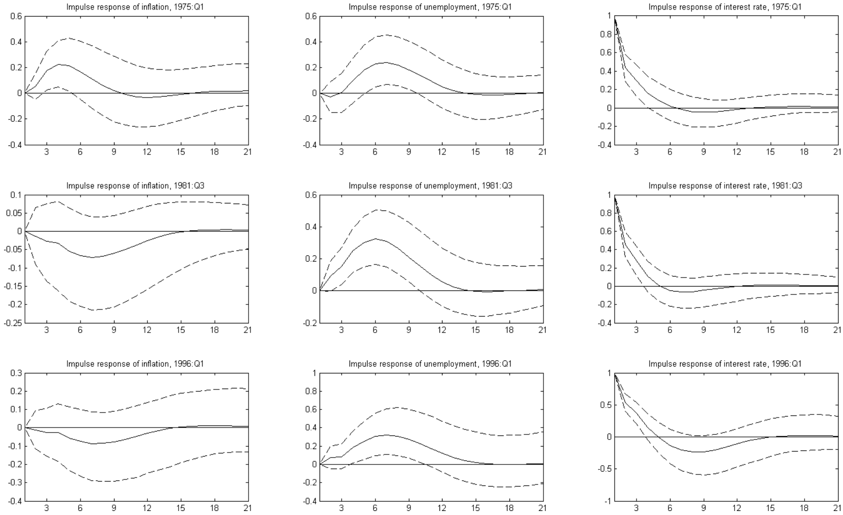
⇒ Illustrate time varying volatility of equations for $\boldsymbol{r}_t$ and factor equations

⇒ Impulse responses at three different time periods

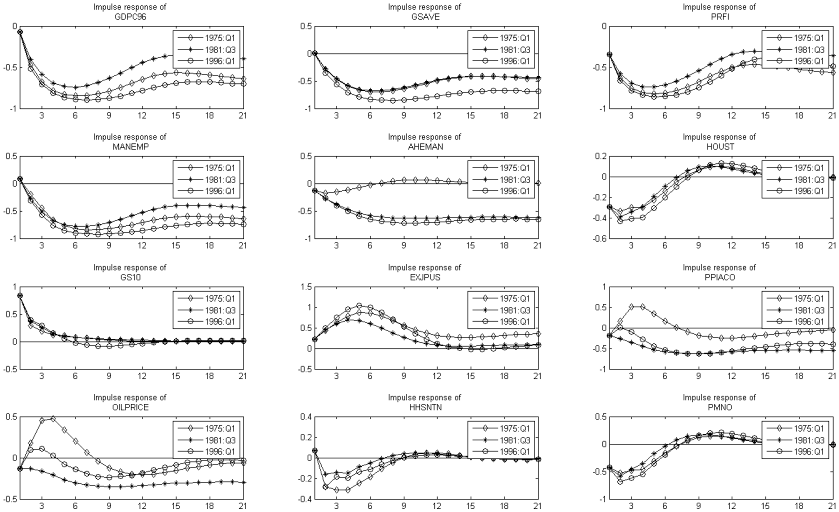
TVP-FAVAR: Volatilities in some key equations



TVP-FAVAR: IRFs of main variables to MP shock



TVP-FAVAR: IRFs of selected other variables



Main take aways of factor models

- $\Rightarrow$ Factor methods are an attractive way of modelling when the number of variables is large
- $\Rightarrow$ DFMs often are good for forecasting
- $\Rightarrow$ FAVARs good for macroeconomic policy (e.g., to do impulse response analysis)
- $\Rightarrow$ Recently TVP versions of these models have been developed
- $\Rightarrow$ Bayesian inference in TVP-FAVAR puts together MCMC algorithm involving blocks from several simple and familiar algorithms

Mixed frequency methods

Introduction to mixed frequency methods

⇒ There are many other popular macroeconomic models which combine state space models with VARs

⇒ Mixed frequency methods commonly used for nowcasting

⇒ Bayesian MCMC methods of earlier lectures can be used

⇒ Any of the VAR priors discussed previously can be used

⇒ I will not discuss prior and computation much (covered in earlier lectures), but focus on the form of the models

Why mixed frequency econometrics?

- ⇒ Mixed frequency VARs (MF-VARs) are enjoying great popularity among policymakers and academics
- ⇒ Produce timely high frequency nowcasts of low frequency variables
- ⇒ For example, GDP growth is quarterly, many other potential predictors are monthly
- ⇒ **Frequency:** Information in monthly data can be used to produce monthly nowcasts/estimates of GDP growth
- ⇒ **Timeliness:** GDP is released with a delay, monthly predictors often released much more quickly (several nowcasts of GDP growth can be produced before initial estimate produced by statistical agencies)

Notation: The VAR (in structural form)

⇒ We work with VARs (some involving latent states) written in the following form:

$$\boldsymbol{A}\boldsymbol{y}_t = \boldsymbol{B}\boldsymbol{y}_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

⇒ $\boldsymbol{A}$ is lower triangular and Σ is diagonal

⇒ Σ contains error variances and $\boldsymbol{A}$ determines covariances (contemporaneous relationships)

⇒ Computational advantage: Writing VAR in this way allows for equation-by-equation estimation (since Σ is diagonal)

⇒ This is equivalent to a reduced form VAR with error covariance matrix $\boldsymbol{A}^{-1}\Sigma\boldsymbol{A}^{-1\prime}$

⇒ Straightforward to add extra lags/exogenous variables, etc.

Notation: Mixed frequency methods

- ⇒ $t = 1, \ldots, T$ is time at the monthly frequency
- ⇒ $\boldsymbol{y}_{m,t}$ is vector of monthly variables (e.g., industrial production)
- ⇒ $\boldsymbol{y}_{q,t}$ are monthly values of variables which are only observed quarterly (e.g., GDP growth)
- ⇒ $\boldsymbol{y}_{m,t}$ are observed, but $\boldsymbol{y}_{q,t}$ are unobserved (latent states)
- ⇒ The statistical agency produces a quarterly figure, $\boldsymbol{y}_{Q,t}$
- ⇒ If $y_{q,t}$ are monthly log differences and $y_{Q,t}$ are quarterly log differences then can show (see Mariano and Murasawa, 2003, *JAE*):

$$\boldsymbol{y}_{Q,t} = \frac{1}{3}\boldsymbol{y}_{q,t} + \frac{2}{3}\boldsymbol{y}_{q,t-1} + \boldsymbol{y}_{q,t-2} + \frac{2}{3}\boldsymbol{y}_{q,t-3} + \frac{1}{3}\boldsymbol{y}_{q,t-4}$$

- ⇒ This is the intertemporal restriction which applies every third month ($\boldsymbol{y}_{Q,t}$ not observed in first and second months of quarter)

Notation: The MF-VAR

⇒ The MF-VAR is the previous VAR but with a different definition for $\boldsymbol{y}_t$

⇒ Dependent variables in VAR: $\boldsymbol{y}_t = (\boldsymbol{y}'_{m,t}, \boldsymbol{y}'_{q,t})'$

⇒ So MF-VAR is VAR, but with some variables being unobserved latent states

⇒ MF-VAR is a state space model:

- → Measurement equations: Intertemporal restriction which relates $\boldsymbol{y}_{Q,t}$ and $\boldsymbol{y}_{q,t}$, and equations which say $\boldsymbol{y}_{m,t}$ are observed
- → VAR provides state equations

Overview of Bayesian inference in the MF-VAR

- $\Rightarrow$ MCMC methods such as Gibbs sampling popular in Bayesian econometrics
- $\Rightarrow$ Draw from conditional posteriors
- $\Rightarrow$ In MF-VAR: Draw from VAR parameters $(\boldsymbol{A}, \boldsymbol{B}, \Sigma)$ conditional on states $\boldsymbol{y}_{q,t}$, then draw from $\boldsymbol{y}_{q,t}$ given $\boldsymbol{A}, \boldsymbol{B}, \Sigma$
- $\Rightarrow$ Draw from VAR parameters: Can use results for any Bayesian VAR (e.g., Minnesota prior, Normal-Wishart, global local shrinkage prior)
- $\Rightarrow$ Draw from states: Standard Bayesian MCMC methods for Normal linear state methods can be used
- $\Rightarrow$ MF-VAR pioneered in Schorfheide and Song, "Real-time forecasting with a mixed-frequency VAR," *JBES*, 2015

Summary

- $\Rightarrow$ A good knowledge of Bayesian state space and multivariate regression methods can get you pretty far
- $\Rightarrow$ Most models used for macroeconomic forecasting and nowcasting are either multivariate regressions (VARs) or state space models or a combination
- $\Rightarrow$ This lecture has been about factor models and mixed frequency methods to provide examples of this combination strategy
- $\Rightarrow$ Many more exist